## SM212, Eigenvalue method in $2 \times 2$ case

PROBLEM: Solve

$$\begin{cases} x' = ax + by, & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

**soln**: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Find the eigenvalues. These are the roots of the characteristic polynomial

  $$p(\lambda) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).$$

  Call them $\lambda_1$, $\lambda_2$ (in any order you like).

  You can use the quadratic formula, for example to get them:

  $$\lambda_1 = \frac{a + d}{2} + \frac{\sqrt{(a + d)^2 - 4(ad - bc)}}{2}, \quad \lambda_2 = \frac{a + d}{2} - \frac{\sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

- Find the eigenvectors. If $b \neq 0$ then you can use the formulas

  $$\vec{v}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$

  In general, you can get them by solving the **eigenvector equation** $A\vec{v} = \lambda\vec{v}$.

- Plug these into the following formulas:

  (a) $\lambda_1 \neq \lambda_2$, real:

  $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\vec{v}_1 \exp(\lambda_1 t) + c_2\vec{v}_2 \exp(\lambda_2 t).$$

  (b) $\lambda_1 = \lambda_2 = \lambda$, real:

  $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\vec{v}_1 \exp(\lambda t) + c_2(\vec{v}_1 t + \vec{p}) \exp(\lambda t),$$

  where $\vec{p}$ is any non-zero vector satisfying $(A - \lambda I)\vec{p} = \vec{v}_1$.

(c) $\lambda_1 = \alpha + i\beta$, complex: write $\vec{v}_1 = \vec{u}_1 + i\vec{u}_2$, where $\vec{u}_1$ and $\vec{u}_2$ are both *real vectors.*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1[\exp(\alpha t)\cos(\beta t)\vec{u}_1 - \exp(\alpha t)\sin(\beta t)\vec{u}_2]$$
$$+c_2[-\exp(\alpha t)\cos(\beta t)\vec{u}_2 - \exp(\alpha t)\sin(\beta t)\vec{u}_1].$$

## Examples

**Example 1** *Solve*

$$x'(t)) = x(t) - y(t), \quad y'(t) = 4x(t) + y(t), \quad x(0) = -1, \quad y(0) = 1.$$

*Let*

$$A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

*and so the characteristc polynomial is*

$$p(x) = \det(A - xI) = x^2 - 2x + 5.$$

*The eigenvalues are*

$$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i,$$

*so $\alpha = 1$ and $\beta = 2$. Eigenvectors $\vec{v}_1, \vec{v}_2$ are given by*

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 2i \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -2i \end{pmatrix},$$

*though we actually only need to know $\vec{v}_1$. The real and imaginary parts of $\vec{v}_1$ are*

$$\vec{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

*The solution is then*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -c_1\exp(t)\cos(2t) + c_2\exp(t)\sin(2t) \\ -2c_1\exp(t)\sin(2t) - 2c_2\exp(t)\cos(2t), \end{pmatrix}$$

*so $x(t) = -c_1\exp(t)\cos(2t) + c_2\exp(t)\sin(2t)$ and $y(t) = -2c_1\exp(t)\sin(2t) - 2c_2\exp(t)\cos(2t)$.*

*Since $x(0) = -1$, we solve to get $c_1 = 1$. Since $y(0) = 1$, we get $c_2 = -1/2$. The solution is: $x(t) = -\exp(t)\cos(2t) - \frac{1}{2}\exp(t)\sin(2t)$ and $y(t) = -2\exp(t)\sin(2t) + \exp(t)\cos(2t)$.*

**Example 2** *Solve*

$$x'(t) = -2x(t) + 3y(t), \quad y'(t) = -3x(t) + 4y(t).$$

*Let*

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$$

*and so the characteristc polynomial is*

$$p(x) = \det(A - xI) = x^2 - 2x + 1.$$

*The eigenvalues are*

$$\lambda_1 = \lambda_2 = 1.$$

*An eigenvector* $\vec{v}_1$ *is given by*

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

*Since we can multiply any eigenvector by a non-zero scalar and get another eigenvector, we shall use instead*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*Let* $\vec{p} = \begin{pmatrix} r \\ s \end{pmatrix}$ *be any non-zero vector satisfying* $(A - \lambda I)\vec{p} = \vec{v}_1$. *This means*

$$\begin{pmatrix} -2-1 & 3 \\ -3 & 4-1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*There are infinitely many possibly solutions but we simply take* $r = 0 ands = 1/3$, *so*

$$\vec{p} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}.$$

*The solution is*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(t) + c_2 ( \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} ) \exp(t),$$

*or* $x(t) = c_1 \exp(t) + c_2 t \exp(t)$ *and* $y(t) = c_1 \exp(t) + \frac{1}{3} c_2 \exp(t) + c_2 t \exp(t)$.